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Chronotopic Lyapunov Analysis: II. Toward a Unified Approach

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From the analyticity properties of the equation governing infinitesimal perturbations, it is conjectured that all types of Lyapunov exponents introduced in spatially extended 1D systems can be derived from a single function that we call the entropy potential. The general consequences of its very existence on the Kolmogorov–Sinai entropy of generic spatiotemporal patterns are discussed.

KEY WORDS: Spatiotemporal chaos; coupled map lattices; entropy potential; comoving Lyapunov exponents.

1. INTRODUCTION

The first part of this work[1] (hereafter referred to as LPT) was devoted to the definition and discussion of the properties of temporal (TLS) and spatial (SLS) Lyapunov spectra of 1D extended dynamical systems. In this second part we first show how the two approaches are deeply related by proving, in some simple cases, and conjecturing, in general, that all stability properties can be derived from a single observable: the *entropy potential*, which is a function of two independent variables, the spatial and the temporal growth rates μ, λ, respectively. Legendre transforms represent the right tool to achieve a complete description of linear stability properties in the space-time plane. In fact, we find that equivalent descriptions can be obtained by choosing any pair of independent variables in the set

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$\{n_\mu, n_\lambda, \mu, \lambda\}$, where n_μ and n_λ are the integrated densities of spatial, resp. temporal, Lyapunov exponents. The corresponding potentials are connected via suitable Legendre transformations involving pairs of conjugated variables.

In the perspective of a complete characterization of space-time chaos, one could also consider the possibility of viewing a generic pattern as being generated along directions other than time and space axes. In fact, once a pattern is given, any direction can, *a priori*, be considered as an appropriate "time" axis. Accordingly, questions can be addressed about the statistical properties of the pattern when viewed in that way. The opportunity to consider generic orientations of the space-time coordinates does not follow simply from an abstract need of completeness; it also represents a first attempt to extend the analysis of chaotic data from time-series to patterns. In such a context, the mere existence of the entropy potential implies that the Kolmogorov-Sinai entropy-density is independent of the orientation. This provides a more sound basis to the conjecture that the entropy density is indeed an intrinsic property of a given pattern.[2]

For the sake of completeness, finally, we recall the last class of exponents introduced to describe convectively unstable states, *comoving* Lyapunov exponents,[3] and their relationship again with SLS and TLS. In particular, we discuss the structure of the spectra in a simple case of a stationary random state.

Let us now briefly introduce the notations with reference to some specific models. Spatiotemporal chaos and instabilities in extended systems have been widely studied with the aid of models of reaction-diffusion processes, whose general 1D form is of the type[4,5]

$$\partial_t \mathbf{y} = \mathbf{F}(\mathbf{y}) + \mathbf{D}\partial_x^2 \mathbf{y} \tag{1}$$

with te state variable $\mathbf{y}(x, t)$ defined on the domain $[0, L]$ (periodic boundary conditions $\mathbf{y}(0, t) = \mathbf{y}(L, t)$ are generally assumed). The nonlinear function $\mathbf{F}$ accounts for the local reaction dynamics, while the diffusion matrix $\mathbf{D}$ represents the strength of the spatial coupling. The introduction of coupled map lattices (CML) has been of great help for understanding the statistical properties of spatio-temporal chaos, especially by means of numerical simulations.

In its standard form[6,7] a CML dynamics reads as

$$y_{n+1}^i = f\left((1-\varepsilon)\, y_n^i + \frac{\varepsilon}{2}\,[y_n^{i-1} + y_n^{i+1}]\right) \tag{2}$$

where i, n being the space, resp. time, indices labelling each variable y_n^i of a lattice of length L (with periodic boundary conditions $y_n^{i+L} = y_n^i$), and

ε gauges the diffusion strength. The function f, mapping a given interval I of the real axis onto itself, simulates a local nonlinear reaction process.

A generalization of model (2) has been proposed[3] to mimic 1D open-flow systems, namely

$$y^i_{n+1} = f((1-\varepsilon)\, y^i_n + \varepsilon[(1-\alpha)\, y^{i-1}_n + \alpha y^{i+1}_n]) \tag{3}$$

The parameter α (bounded between 0 and 1) accounts for the possibility of an asymmetric coupling, corresponding to first order derivatives in the continuum limit.

The present paper is organized as follows. In Section 2 we introduce the entropy potential and derive its explicit expression in some simple cases. Section 3 is devoted to a discussion of some general consequences of the existence of a potential on the Kolmogorov-Sinai entropy. Comoving exponents are reviewed in Section 4 within the framework introduced in this paper. Some conclusive remarks are finally reported in Section 5.

2. ENTROPY POTENTIAL

The simplest context, where a discussion on the entropy potential can be set in, is provided by the linear diffusion equation for the field $u(x, t)$

$$\partial_t u = \gamma u + D\partial_x^2 u \tag{4}$$

which can be interpreted as the linearization of (the scalar version of) Eq. (1) around a uniform stationary solution $y(x, t) = \text{const}$. The linear stability analysis amounts to assuming a perturbation of the form

$$u(x, t) \sim \exp(\tilde{\mu} x + \tilde{\lambda} t) \tag{5}$$

where $\tilde{\lambda} = \lambda + i\omega$ and $\tilde{\mu} = \mu + ik$ are complex numbers the real parts of which denote temporal and spatial Lyapunov exponents. Substituting Eq. (5) in Eq. (4) we obtain

$$\tilde{\lambda} = \gamma + D\tilde{\mu}^2 \tag{6}$$

By separating real and imaginary parts, we get

$$\begin{aligned} \lambda(\omega, k) &= \gamma + (\omega^2 - 4D^2k^4)/(4Dk^2) \\ \mu(\omega, k) &= \omega/(2Dk) \end{aligned} \tag{7}$$

As already discussed in LPT, ω and k play the same role as the integrated densities n_μ and n_λ. This is a consequence of the fact that the Lyapunov vectors are simply the Fourier modes, and the eigenvalues are naturally ordered by the corresponding wavenumbers. Such integrated densities can be explicitly obtained by inverting Eqs. (7),

$$n_\lambda \equiv k = \sqrt{\mu^2 - \frac{\lambda - \gamma}{D}}$$
$$n_\mu \equiv -\omega = -2D\mu \sqrt{\mu^2 - \frac{\lambda - \gamma}{D}} \tag{8}$$

The minus sign in the definition of n_μ is just a matter of convention: we adopt this choice for consistency reasons with LPT.

The above sets of Eqs. (7) and (8) stress, in a particular instance, the general observation reported in LPT that either the pair (n_μ, n_λ) or (μ, λ) suffices to identify a given perturbation, the remaining two variables being determined from the Lyapunov spectra. However, any two items in the set $\{\mu, \lambda, n_\mu, n_\lambda\}$ can be chosen to be the independent variables. The above two choices are preferable for symmetry reasons; however, the pairs (μ, n_λ) and (λ, n_μ) turn out to be the best ones for the identification of a single function, the entropy potential, which determines all stability properties.

In fact, as it is clear from Eq. (6), we can condense the two real functions needed for a complete characterization of the stability properties into a single analytic complex expression. Now, the mere circumstance that $\lambda(\mu, n_\lambda)$ and $n_\mu(\mu, n_\lambda)$ are the real and imaginary parts of the analytic function $\tilde{\lambda}^*(\tilde{\mu})$ has an immediate and important consequence[5]: Cauchy-Riemann conditions are satisfied and it is possible to write λ and n_μ as partial derivatives of the same real function,

$$\frac{\partial \Psi}{\partial n_\lambda} = \lambda$$
$$\frac{\partial \Psi}{\partial \mu} = -n_\mu \tag{9}$$

where Ψ is the imaginary part of the formal integral $\tilde{\Psi}$ of $\tilde{\lambda}$ with respect to $\tilde{\mu}$. Equivalently, one might call into play the real part of $\tilde{\Psi}$, as it is known that the latter contains the same amount of information.

In the case under investigation, we find

$$\Psi(\mu, n_\lambda) = n_\lambda(\gamma + D\mu^2) - \frac{D}{3} n_\lambda^3 \tag{10}$$

[5] The reference to the complex conjugate variable again follows from the convention adopted for n_μ.

which, together with Eq. (9), provides a complete characterization of the system.

Another, less trivial, example where the linearized problem leads to an analytic function for the eigenvalues is the 1D complex Ginzburg-Landau equation[4,5]

$$\partial_t A = (1 + ic_1)\, \partial_x^2 A + A - (1 - ic_3)\, A\, |A|^2 \tag{11}$$

where $A(x, t)$ is a complex field and c_1 and c_3 are real positive parameters. The stability of the "phase winding" solutions $A(x, t) = A_0 \exp(i(vx - \omega_0 t))$, with $A_0 = \sqrt{1 - v^2}$ and $\omega_0 = -c_3 + (c_1 + c_3)\, v^2$, are ruled by the following equation for the (complex) perturbation $u(x, t)$

$$\partial_t u = (1 + ic_1)(\partial_x^2 u + 2i\partial_x u) - (1 - ic_3)(1 - v^2)(u + u^*) \tag{12}$$

together with its complex conjugate for u^* considered as an independent variable. The eigenvalue problem is solved assuming again

$$u(x, t) = u_0 \exp(\tilde{\mu} x + \tilde{\lambda} t); \qquad u^*(x, t) = u_0^* \exp(\tilde{\mu} x + \tilde{\lambda} t) \tag{13}$$

and equating to zero the determinant of the resulting linear system, to get the analytic (implicit) relation between $\tilde{\lambda}$ and $\tilde{\mu}$

$$(\tilde{\lambda} + 1 - v^2 - \tilde{\mu}^2 + 2c_1 v\tilde{\mu})^2 + (c_1\tilde{\mu}^2 + 2v\tilde{\mu} + (1 - v^2)\, c_3)^2 = (1 + c_3^2)(1 - v^2)^2 \tag{14}$$

which is analogous to Eq. (6).

On the basis of the examples discussed here and in the Appendix, one can convince himself that the analyticity of the eigenvalue equation appears to be very general. Periodicity in time simply leads to multiply several r.h.s.'s all depending on $\tilde{\mu}$, while periodicity in space requires distinguishing between different sites on the lattice. In the latter case, the equivalent of Eq. (A7) is obtained by equating to zero a suitable determinant, where the only variable is $\tilde{\mu}$. There is no reason to expect that different conditions should hold in aperiodic regimes.

The above approach is, in some sense, a generalization of dispersion relations which are normally introduced for the characterization of elliptic equations. In that case, the only acceptable linear solutions are propagating plane waves, that is $\lambda = \mu = 0$ for (almost) all wavenumbers. From our point of view, this implies a strong simplification since the mutual relationships among $\{n_\lambda, n_\mu, \lambda, \mu\}$ reduce to the link between spatial and temporal wavenumbers. Moreover, it is obvious that, because of the degeneracy, not all representations are equivalent (in particular, the (μ, λ) plane is totally useless).

We must stress that the methodology that we are trying to develop in these two papers applies to general systems where propagation coexists with amplification (or damping). This is by no means a limitation, as all models introduced for the characterization of space-time chaos are in this class.

However, the most serious obstacle to a rigorous proof of the general validity of Eq. (9) is represented by the identification of the integrated densities n_μ, n_λ with the wavenumbers ω and k, respectively. In the presence of spatial disorder, the Lyapunov vectors are no longer Fourier modes: one can at most determine an average wavenumber by counting the number of nodes in the eigenfunctions. This is not a problem in the absence of temporal disorder, when the node theorem applies.[8] However, in more general cases, it is no longer possible to speak of eigenfunctions and we are not aware of any generalization to overcome the difficulty. For such a reason, we have performed some direct numerical check to verify the correctness of our conjectures.

Before discussing numerical simulations, let us come back to the problem of the representation. In the above part, we have seen that the choice of the pair of independent variables (μ, n_λ) was very fruitful for the identification of a potential. However, the asymmetry of such a choice calls for transferring the above result in either representation proposed in LPT. This step can be easily done with the help of Legendre transforms. We discuss the transformation to the plane (μ, λ), any other transformation being a straightforward generalization of the same procedure.

From the first of Eq. (9), we see that λ and n_λ can, indeed, be considered as conjugate variables in a Legendre transform involving Ψ. The conjugate potential is naturally

$$\Phi \equiv \lambda n_\lambda - \Psi \tag{15}$$

It is easily seen that in the new representation, the following relations hold

$$\begin{aligned} \partial_\lambda \Phi &= n_\lambda \\ \partial_\mu \Phi &= n_\mu \end{aligned} \tag{16}$$

Accordingly, the potential Φ is the appropriate function which allows determining the two integrated densities in the symmetric representation (λ, μ). We call Φ the entropy potential since it coincides with the Kolmogorov-Sinai entropy density along a suitable line (see Section 3).

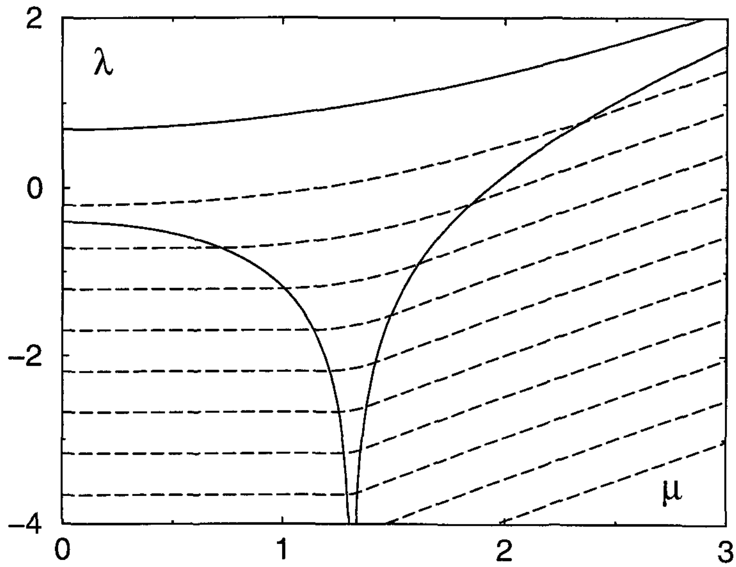

Fig. 1. Contour plot of the entropy potential Φ for a homogeneous chain ($r = 2$, $\varepsilon = 1/3$).

In Fig. 1, we present a numerical reconstruction of $\Phi(\mu, \lambda)$ in terms of its contour levels for the homogeneous, CML, namely model (2) with $f(x) = rx$ (mod 1). This is one of the models discussed in the Appendix for which it can be proved that the entropy potential does exist and, even more, an integral expression for Φ is available. However, because of the lack of an explicit solution, we have found more convenient to determine $\Phi(\mu, \lambda)$ by integrating (n_μ, n_λ) (see Eqs. (19) and (20) in LPT) first along the horizontal path $\mu = 0$ (from the origin to the prescribed λ-value) and then vertically to (μ, λ). The entropy potential is obviously known up to an additive arbitrary constant that we have fixed by imposing that the value

Table I. Entropy Potential Φ Computed by Integrating Along Two Different Paths in Two Different Points of the (μ, λ) Plane[a]

Path	Integral	Path	Integral	Φ
$(0, 0) \to (0, 3)$	0.1011	$(0, 3) \to (2, 3)$	−0.3883	−0.2872
$(0, 0) \to (2, 0)$	−0.5274	$(2, 0) \to (2, 3)$	0.2406	−0.2868
$(0, 0) \to (0, 0.8)$	0.1069	$(0, 0.8) \to (3, 0.8)$	−0.8036	−0.6967
$(0, 0) \to (3, 0)$	−1.4972	$(3, 0) \to (3, 0.8)$	0.8	−0.6972

[a] The difference is definitely smaller than the statistical error ($\approx 10^{-3}$).

attained on the upper border is equal to zero. The potential increases monotonically from top to bottom. It is clear that outside the allowed region delimited by the solid curves, Φ is a linear function of μ and λ.

The structure of the potential does not substantially change for more general CMLs. We have tested Eq. (16) for a lattice of logistic maps ($f(x) = 4x(1-x)$) with $\varepsilon = 1/3$ by integrating along two different paths in the (μ, λ) plane (see Table I). The difference is so small that we can confirm that the relations are valid, within the numerical error.

3. KOLMOGOROV-SINAI ENTROPY

The present section is devoted to discuss the consequence of the existence of the entropy potential on the Kolmogorov-Sinai entropy H_{KS} which is a measure of the information-production rate during a chaotic evolution. An estimate of H_{KS} is given by the Pesin formula[9] as the sum H_λ of the positive Lyapunov exponents. While it is rigorously proven that $H_{KS} \leqslant H_\lambda$, numerical simulations indicate that, in general, an equality holds.

In spatially extended systems, H_{KS} is believed to be proportional to the system size.[10] For this reason, it is convenient to introduce the entropy density h_λ which, in the thermodynamic limit, is the integral of the positive part of the Lyapunov spectrum,

$$h_\lambda = \int_0^{\lambda_{\max}} n_\lambda(\mu = 0, \lambda)\, d\lambda \tag{17}$$

where $\lambda_{\max}$ is the standard maximum Lyapunov exponent.

One can naturally extend the above formula to the case of SLS, defining in a similar way the spatial entropy density h_μ. It is important to notice that both quantities have the same physical dimensions, as they are measured in [*bits/lt*]. Numerical simulations performed with different CML models indicate that $h_\lambda < h_\mu$. This can be explained by the following argument. The patterns obtained asymptotically by iterating the model in the original reference frame are, in general, unstable if generated along the spatial direction.[11] In other words, the spatiotemporal attractor is a (strange) repellor of the spatial dynamics. Accordingly, part of the local instability accounted for by the sum of positive spatial Lyapunov exponents is turned into a contribution to the escape rate from the repellor,[12] and h_μ must be larger than the entropy h_λ of the original pattern. In continuous models this inequality is brought to the extreme case, as h_μ is infinite.

In the context of low-dimensional chaos, one is interested in understanding whether a given irregular signal originates from a few nonlinearly coupled degrees of freedom. In the context of space-time chaos, patterns are the object of investigation. At variance with time-series that can be analyzed either by moving forward or backward in time, in the case of patterns, the identification of the most appropriate spatial and temporal directions is a new and unavoidable element of the game. To this aim, the chronotopic formalism can be extended to consider an arbitrary orientation of the "temporal" axis, identified by the unit vector $\vec{u} = (\sin\theta, \cos\theta)$. This results in introducing a new class $\eta(n_\eta)$ of Lyapunov spectra.[2,13] In the same way as the TLS and the SLS are defined by moving along the coordinate axes in the (μ, λ) plane, one can show that the η-spectra correspond to moving along the line $\mathscr{L}$ defined by $\lambda = \mu \tan\theta$.[13]

Here, we limit ourselves to discuss some implications of the above ideas on the concept of entropy density for patterns. Actually, the very existence of the entropy potential Φ implies that the Lyapunov spectrum is given by

$$n_\eta(\theta, \eta) = \vec{u} \cdot \nabla\Phi \tag{18}$$

where $\nabla = (\partial_\mu, \partial_\lambda)$ is the gradient in the (μ, λ) plane, and the r.h.s. of the above formula is evaluated along the line $\mathscr{L}$.

It is natural to extend the definition (17) of entropy as

$$h_\eta = \int_0^{\eta_{\max}} n_\eta(\theta, \eta)\, d\eta \tag{19}$$

where the integral is performed along the line $\mathscr{L}$ and $\eta_{\max}$ is the maximum value of η which is reached in the intersection point between $\mathscr{L}$ and $\mathscr{D}$. In the limit $\theta = 0$, the above equation reduces to the previous definition of h_λ, while for $\theta = \pi/2$ it reduces to h_μ.

Again, the existence of an entropy potential implies that

$$h_\eta(\theta) = \begin{cases} h_\lambda & |\theta| < \theta^* \\ h_\mu & |\theta| \geqslant \theta^* \end{cases} \tag{20}$$

where θ^* is the value for which the line $\mathscr{L}$ is tangent to $\mathscr{D}$. This can be shown by first rewriting Eq. (19) as an integral in the plane (n_μ, n_λ),

$$h_\eta = \int_A^B \eta(n_\mu, n_\lambda)\, dn_\eta \tag{21}$$

where $A = (0, n_\lambda(0, 0))$, while B is equal to either $(0, 0)$ if $|\theta| < \theta^*$, or, otherwise, to $(-1, 1)$.

If we further notice that the integrand in Eq. (21) is the gradient of the potential

$$\tilde{\Phi} = \lambda n_\lambda + \mu n_\mu - \Phi \tag{22}$$

it is mathematically obvious why Eq. (20) holds.

The independency of h_η of θ has also a physical interpretation. The Kolmogorov-Sinai entropy density is, in fact, the amount of information needed to characterize a space-time pattern divided by the product of its temporal duration times the spatial extension, i.e., divided by the "area" of a space-time domain. Therefore, if the information flow through the boundaries of such domain is negligible,[10] we expect it to be independent of the way the temporal axis is oriented in the plane.

4. COMOVING EXPONENTS

Another class of indicators, introduced to describe convective instabilities in open-flow systems, consists of the so-called comoving or velocity-dependent Lyapunov exponents.[3] They quantify the growth rate of a localized disturbance in a reference frame moving with constant velocity V. Given an initial perturbation $u(x, 0)$ which is different from zero only within the spatial interval $[-L_0/2, L_0/2]$, numerical analyses indicate

$$u(x, t) \sim \exp(\Lambda(x/t)\, t) \tag{23}$$

for t sufficiently large. Equation (23) defines the comoving Lyapunov exponent Λ as a function of $V = x/t$. The initial width L_0 of the disturbance is not a relevant parameter, since a generic perturbation grows with the maximum rate.[6]

The definition of Λ can be extended to a whole spectrum of comoving exponents by looking not just at the local amplitude of the perturbation but also at its shape.[15] Since the physical meaning of the rest of the spectrum is still questionable, in the following we limit ourselves to discuss the maximum.

As a matter of fact, the limit $t \to \infty$ (required by a meaningful definition of an asymptotic rate) implies the infinite-size limit. Therefore, one must carefully keep under control the system size, when longer times are considered. This is perhaps the most severe limitation against an accurate direct measurement of Λ.

[6] In the particular case of a δ-like initial profile, the definition of local Lyapunov exponent introduced in ref. 14 is recovered.

It can be easily shown that $\Lambda(V)$ is connected with the maximal temporal Lyapunov exponent $\lambda_{\max}(\mu)$ by a Legendre-type transformation.[11] Equation (23) implies that the perturbation has a locally exponential profile with a rate

$$\mu = \frac{d\Lambda(V)}{dV} \tag{24}$$

in the point $x = Vt$. On the other hand, we know that such a profile evolves as

$$u(Vt, t) \sim \exp[(\lambda_{\max}(\mu) + \mu V)\, t] \tag{25}$$

By combining Eqs. (23) and (25), we obtain

$$\Lambda(V) = \lambda_{\max}(\mu) + \mu \frac{d\lambda_{\max}(\mu)}{d\mu} \tag{26}$$

which, together with Eq. (24) can be interpreted as a Legendre transform from the pair (Λ, V) to the pair $(\lambda_{\max}, \mu)$. The inverse transform reveals the further constraint

$$V = \frac{d\lambda_{\max}(\mu)}{d\mu} \tag{27}$$

Equation (26) states that $\Lambda(V)$ is the growth rate of an exponentially localized perturbation with a given μ value as determined from the condition Eq. (24). However, the perturbation itself propagates with yet another velocity, $\tilde{V}(\mu) = \lambda_{\max}(\mu)/\mu$.[16] As a matter of fact, $\tilde{V}(\mu)$ and V correspond to phase and group velocities respectively for propagating waves in linear dispersive media. In particular, the "phase" velocity $\tilde{V}(\mu)$ can be larger than the "light" velocity (which is equal to 1 in CML with nearest neighbour coupling), while V is bounded to be smaller.

A simple geometrical interpretation of the above Legendre transformations can be given with reference to the (μ, λ) plane. The comoving Lyapunov exponent $\Lambda(V)$ is the intercept of the λ axis with the straight line of slope V, tangent to the upper branch of $\mathscr{D}$. If the system is chaotic, such an intersection remains positive for $V \leqslant V_*$, where $V_* = \min(\tilde{V}(\mu))$ can be interpreted as the propagation velocity of initially localized disturbances[16]; faster perturbations are exponentially damped.

Whenever a Legendre transform comes into play, some attention must be payed to the concavity of the functions involved in the transformation.

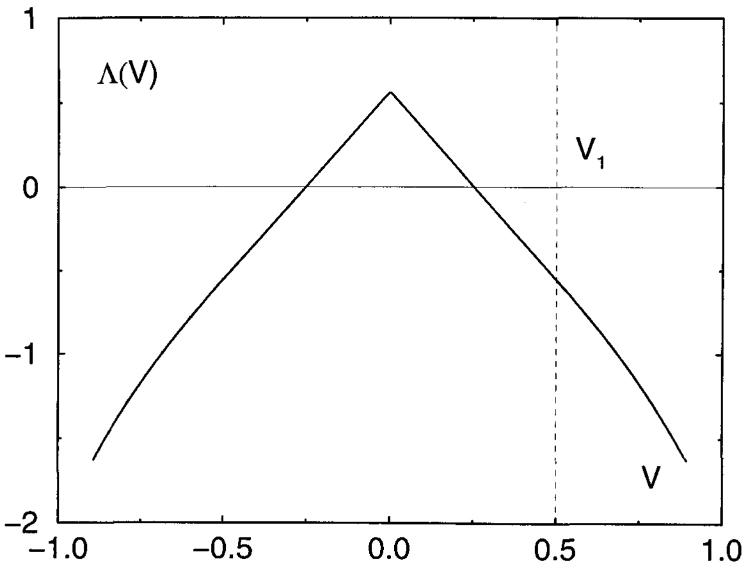

Fig. 2. Maximum comoving Lyapunov exponent $\Lambda(V)$ for a frozen random pattern obtained as Legendre transform versus V (for comparison see also Fig. 6 in LPT). The vertical line indicates the position of the critical velocity V_1 (see the text for definition).

In the present context, this is the case of frozen random patterns where the border of the allowed region exhibits a change of concavity at $|\mu| = \mu_1$ (see Fig. 6 of LPT). This implies that for $|\mu| < \mu_1$, the maximal temporal exponent is constant and equal to $\lambda_{\max}(\mu_1)$. The corresponding "phase transition" reflects itself in the onset of a linear dependence of the comoving Lyapunov exponent on the velocity for $|V|$ smaller than some value V_1,

$$\Lambda(V) = \lambda_{\max}(\mu_1) - \mu_1 V \tag{28}$$

This is evident in Fig. 2, where the whole set of Λ values is reported.

5. CONCLUSIONS

In the present paper we have shown that all instability properties of 1D chaotic systems can be derived from a suitable entropy potential expressed as a function of any pair of variables in the set $\{\mu, \lambda, n_\mu, n_\lambda\}$. The most appropriate representation depends on the problem under investigation. For instance, the properties of Kolmogorov-Sinai entropy are more naturally described with reference to (n_μ, n_λ). This is analogous to standard thermodynamics, where several potentials (Gibbs, Helmholtz, etc.) are introduced to cope with different physical conditions.

The very notion of entropy potential implies general relations among the classes of Lyapunov exponents introduced and discussed here and in LPT, namely spatial, temporal and comoving exponents. Another remarkable consequence of the existence of an entropy potential is the independency of h_η on the propagation direction in the space-time plane. Accordingly, the Kolmogorov-Sinai entropy density can be considered as a super-invariant dynamical indicator of a given pattern.

A further remark concerns the space dimensionality. The existence of the entropy potential appear to stem from the analyticity of the complex dispersion relations which, in turn, is peculiar of 1D systems.

Although our theoretical construction is mainly based on numerical simulations and rigorous proofs in extremely simple models, we suspect that the existence of the entropy potential follows from some fundamental principle. Anyhow, we hope that the results presented in this paper will stimulate further studies along these lines.

APPENDIX A

The crucial point in justifying the existence of the entropy potential is the analytic structure of the eigenvalue equation stemming from the linearized dynamics. To support the generality of this statement we consider in this Appendix two more examples, namely the linear stability analysis of homogeneous solutions both of the 1D wave equation and of CML.

The wave equation

$$\partial_t^2 u = -m^2 u + \partial_x^2 u \tag{A1}$$

is the conservative analogous of Eq. (4) (m is a real parameter) and can be treated in a similar way, obtaining

$$\tilde{\lambda}^2 = \tilde{\mu}^2 - m^2 \tag{A2}$$

The above expression justifies *per se* the existence of the entropy potential. Incidentally, notice that the Hamiltonian nature of Eq. (A1) implies the degeneracy of the standard TLS in zero, since the uniform solution is an elliptic fixed point. The entropy potential is determined as the real or, equivalently, the imaginary part of the formal integral

$$\tilde{\Psi}(\tilde{\mu}) = \int \tilde{\lambda}\, d\tilde{\mu} = \frac{1}{2}\left[\tilde{\mu}^2 - m^2 \cosh^{-1}\left(\frac{\tilde{\mu}}{m}\right)\right] \tag{A3}$$

This can be verified in the limit of a "weak" instability $m \to 0$, when Eq. (A3) approximately reads as

$$\tilde{\Psi}(\tilde{\mu}) \approx \frac{1}{2}\left[\tilde{\mu}^2 - m^2 \log\left(\frac{\tilde{\mu}}{m}\right)\right] \tag{A4}$$

By also expanding to the lowest order in m the expressions of λ and ω determined by Eq. (A2), we obtain

$$\begin{aligned} \lambda(\mu, k) &\approx |\mu|\left(1 - \frac{1}{2}\frac{m^2}{\mu^2 + k^2}\right) \\ \omega(\mu, k) &\approx k\left(1 + \frac{1}{2}\frac{m^2}{\mu^2 + k^2}\right) \end{aligned} \tag{A5}$$

It is straightforward to verify that

$$\begin{aligned} \partial_k \operatorname{Re} \tilde{\Psi} &= -\partial_\mu \operatorname{Im} \tilde{\Psi} = \omega \\ \partial_\mu \operatorname{Re} \tilde{\Psi} &= \partial_k \operatorname{Im} \tilde{\Psi} = \lambda \end{aligned} \tag{A6}$$

For homogeneous solutions of CML models, we obtain

$$e^{\tilde{\lambda}} = r[(1-\varepsilon) + \varepsilon \cosh \tilde{\mu}] \tag{A7}$$

where r is the multiplier. Unfortunately, in this case it is not possible to write down an explicit expression for the integral $\tilde{\Psi}$ for generic parameter values. We limit ourselves to discuss the problem in the limit of a small coupling, i.e., $\varepsilon \to 0$. Expansion of (A7) to the first order in ε, yields

$$\tilde{\Psi}(\tilde{\mu}) \approx (\log r - \varepsilon)\, \tilde{\mu} + \varepsilon \sinh \tilde{\mu} \tag{A8}$$

and

$$\begin{aligned} \lambda(\mu, n_\lambda) &\approx \log r - \varepsilon(1 - \cos k \cosh \mu) \\ n_\mu(\mu, n_\lambda) &\approx \varepsilon \sin k \sinh \mu \end{aligned} \tag{A9}$$

which should be compared with the corresponding expressions obtained by expanding to first order in ε Eqs. (16) and (20) of LPT. Moreover, one can verify that the relations analogous to Eqs. (A6) hold also in the present example.

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